

A selfadjoint variant of the time operator

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Abstract

We study the selfadjoint time operator recently constructed by one of the authors. We will show that this time operator must be interpreted as a “selfadjoint variant” of the time operator.

1 Introduction

Ever since Pauli [1] proved that there cannot exist a selfadjoint time operator T that commutes canonically with a bounded-from-below Hamiltonian H , there have been many attempts to construct meaningful time operators in quantum mechanics.

One way to circumvent Pauli’s objection is to work directly with the non-selfadjoint T by way of Positive Operator Valued Measures (POVMs), see e.g. [2–5] and especially the reviews in [6]. Another way is to manipulate the original time operator T until we obtain a meaningful, selfadjoint time operator. The resulting operators are called “selfadjoint variants” of the time operator [7]. In this second category, we find the “selfadjoint variants” of the time-of-arrival operator of Razavi [8], that of Grot *et al.* [9] (see also [10]), that of Kijowski [11, 12], and that of Galapon *et al.* [13]. Although these operators are all selfadjoint, they lack other desirable properties such as covariance [7] or a canonical commutation relation with H . There are time operators, such as the dwell time [14] or the time delay [15], that are selfadjoint and commute with the Hamiltonian, and therefore Pauli’s theorem does not apply to them. In the relativistic domain, it has been recently reported that the time-of-arrival operator has self-adjoint extensions [16].

The efforts to construct time operators parallel those to construct phase operators. It is known that there cannot exist a selfadjoint phase operator that canonically commutes with the number operator [17]. Similarly to the time operator, one has to either work with the original phase operator using POVMs [18] or construct selfadjoint variants of it, see e.g. [19, 20].

In a recent paper [21], one of us has proposed a selfadjoint time operator, denoted $T_{\sqrt{}}$. We wish here to find the place of $T_{\sqrt{}}$ among the other time observables. Its properties,

especially the fact that $T_\sqrt{}$ does not canonically commute with the Hamiltonian, will lead us to conclude that $T_\sqrt{}$ is another “selfadjoint variant” of the time operator. We will also see that incorporating $T_\sqrt{}$ into the algebra of observables leads to a variant of the Heisenberg algebra.

2 Construction of $T_\sqrt{}$

Let us first summarize the construction of $T_\sqrt{}$ following [21]. We use the holomorphic Fourier transformation (HFT)

$$\begin{aligned}\varphi(t) &= \frac{1}{\sqrt{2\pi\hbar}} \int_0^\infty dE f(E) e^{\frac{i}{\hbar}Et} \\ f(E) &= \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^\infty dt \varphi(t) e^{-\frac{i}{\hbar}Et}\end{aligned}\tag{1}$$

where t is a complex variable defined on the upper half plane \mathbb{H} , and E is a real variable defined on the positive axis $[0, \infty)$. We can promote t and E to quantum operators T and H by defining

$$(Hf)(E) := E f(E), \quad (Tf)(E) := i\hbar \frac{df}{dE}.\tag{2}$$

The HFT (1) provides a conjugate representation of Eq. (2):

$$(H\varphi)(t) = -i\hbar \frac{d\varphi}{dt}, \quad (T\varphi)(t) = t \varphi(t).\tag{3}$$

Obviously, T and H satisfy Heisenberg’s commutation relation:

$$[T, H] = i\hbar \mathbf{1}.\tag{4}$$

Equation (4) is valid in the intersection of the domains of T and H , which we are going to specify now.

The domain of H is

$$D(H) = \{f \in L^2([0, \infty), dE) : Ef \in L^2([0, \infty), dE)\},\tag{5}$$

which is dense in $L^2([0, \infty), dE)$. On $D(H)$, the operator H is symmetric,

$$\langle g|Hf \rangle = \langle Hg|f \rangle.\tag{6}$$

The defect indices d_\pm of H are equal to zero, and therefore H is selfadjoint [22]. Its spectrum is the positive real line,

$$\sigma(H) = [0, \infty).\tag{7}$$

The properties of T are subtler. A straightforward calculation yields

$$\langle g|Tf \rangle = i\hbar f(0)g^*(0) + \langle Tg|f \rangle,\tag{8}$$

so T is symmetric on the domain

$$D(T) = \{f \in L^2([0, \infty), dE) : f \in AC([0, \infty)), f' \in L^2([0, \infty), dE), f(0) = 0\},\tag{9}$$

where $AC([0, \infty))$ denotes the space of absolutely continuous functions on the positive real line. The adjoint T^\dagger also acts as $i\hbar d/dt$, but on the following domain:

$$D(T^\dagger) = \{f \in L^2([0, \infty), dE) : f \in AC([0, \infty)) \text{ , } f' \in L^2([0, \infty), dE)\} \text{ ,} \quad (10)$$

where the condition $f(0) = 0$ has been lifted. The defect indices of T are $d_+(T) = 0$, $d_-(T) = 1$. Because a symmetric operator has selfadjoint extensions if and only if its defect indices are equal, we conclude that T admits no selfadjoint extension. Since T is not selfadjoint, the spectrum of T cannot be real and in fact is

$$\sigma(T) = \mathbb{H} \cup \mathbb{R} \text{ .} \quad (11)$$

Even though T is not selfadjoint and its spectrum includes complex numbers, a way was found in [21] to construct a selfadjoint time operator with real spectrum out of T . In order to do so, we need to construct first the operator T^2 . This operator acts as

$$T^2 = -\hbar^2 \frac{d^2}{dE^2} \text{ .} \quad (12)$$

The operator T^2 is symmetric, and its defect indices are $d_-(T^2) = d_+(T^2) = 1$. Hence, T^2 has infinitely many selfadjoint extensions. The selfadjoint extension used in [21] has the following domain:

$$D(T_F^2) = \{f \in L^2([0, \infty), dE) : f(0) = 0 \text{ , } f \in AC^2([0, \infty)) \text{ , } f'' \in L^2([0, \infty), dE)\} \text{ ,} \quad (13)$$

where $AC^2([0, \infty))$ stands for the space of functions whose first derivative is absolutely continuous. The operator (12) acting on the domain (13) is simply the Friedrichs extension [22], and its spectrum coincides with the positive real line:

$$\sigma(T_F^2) = [0, \infty) \text{ .} \quad (14)$$

The crucial point is that the square root of the Friedrichs extension allows us to define a selfadjoint time operator:

$$T_\surd := +\sqrt{T_F^2} \text{ .} \quad (15)$$

Because it is the square root of a selfadjoint operator, T_\surd is selfadjoint. In particular, its spectrum is real and coincides with the positive real line:

$$\sigma(T_\surd) = [0, \infty) \text{ .} \quad (16)$$

3 Properties of T_\surd

The properties of T_\surd are determined by those of T_F^2 through the spectral theorems. Thus, in order to obtain the properties of T_\surd , we need first to obtain the properties of T_F^2 . From Eq. (15), one may naively think that T_\surd is not only selfadjoint but also satisfies all the nice properties of T such as Eq. (4). On the contrary, we will see that the properties of T_\surd differ drastically from those of T , the reason being that the operation of taking the Friedrichs extension does not commute with the operation of taking the square root.

In the energy representation, the eigenfunctions of T_F^2 are given by

$$\langle E|t\rangle = \sqrt{\frac{2}{\pi\hbar}} \sin(Et/\hbar). \quad (17)$$

These eigenfunctions are delta-normalized,

$$\int_0^\infty dE \langle t|E\rangle \langle E|t'\rangle = \delta(t-t'), \quad (18)$$

and their corresponding eigenvalue is t^2 ,

$$T_F^2|t\rangle = t^2|t\rangle. \quad (19)$$

Thus, the spectral representation of T_F^2 is

$$T_F^2 = \int_0^\infty dt t^2 |t\rangle \langle t|. \quad (20)$$

By contrast to the original time operator $T = i\hbar d/dE$, the eigenfunctions of T_F^2 can be delta-normalized, see Eq. (18), and therefore one can construct a time representation associated with T_F^2 . The unitary operator U that transforms from the energy representation into the time representation reads

$$(Uf)(t) = \int_0^\infty dE \sqrt{\frac{2}{\pi\hbar}} \sin(Et/\hbar) f(E). \quad (21)$$

The operator U brings the original Hilbert space $L^2([0, \infty), dE)$ onto the Hilbert space of the time representation $L^2([0, \infty), dt)$. In the time representation, T_F^2 acts simply as multiplication by t^2 , whereas the squared Hamiltonian acts as

$$H^2 = -\hbar^2 \frac{d^2}{dt^2}, \quad (22)$$

as can be easily seen by using Eq. (21). Note that the Hamiltonian H does not have a simple form in the time representation. In particular, H does not act as $i\hbar d/dt$.

The time operator $T_\sqrt{}$ is unambiguously defined by the spectral theorems. It has the same eigenfunctions as T_F^2 , but the corresponding eigenvalue is t :

$$T_\sqrt{}|t\rangle = t|t\rangle, \quad (23)$$

that is, its spectral representation is

$$T_\sqrt{} = \int_0^\infty dt t |t\rangle \langle t|. \quad (24)$$

Note that, in particular, the eigenfunctions of $T_\sqrt{}$ are the sine functions (17) rather than the original eigenfunctions $e^{iEt/\hbar}$ of the operator T , and that $T_\sqrt{}$ acts as multiplication by t in the time representation. Note also that the time representation associated with $T_\sqrt{}$ is a well-defined representation, not just a POVM as is the case of the original time operator T .

Although selfadjoint, T_{\checkmark} does not canonically commute with the Hamiltonian, because T_{\checkmark} does not act as $i\hbar d/dE$ in the energy representation (or, what is the same, H does not act as $i\hbar d/dt$ in the time representation). Thus, even though we generated a selfadjoint operator from our original time operator T , the resulting T_{\checkmark} does not satisfy Eq. (4). Hence T_{\checkmark} is a “selfadjoint variant” of the time operator.

Since it is selfadjoint, T_{\checkmark} is an observable quantity according to the standard rules of quantum mechanics. We can therefore include it in the algebra of observables. In this paper, the other observable we are considering is the Hamiltonian, and therefore the simplest algebra one can think of is that generated by T_{\checkmark} , H and the identity 1. However, because T_{\checkmark} and H do not commute canonically, the commutator

$$I = [T_{\checkmark}, H] \quad (25)$$

is not given in terms of the generators T_{\checkmark} , H and 1. In order to close the algebra, we must therefore include I as one of the generators:

$$\mathcal{A} \equiv \{T_{\checkmark}, H, I, 1\}. \quad (26)$$

If we had a Heisenberg algebra, then I would equal $i\hbar 1$, and it is in this sense that the algebra \mathcal{A} is a variant of the Heisenberg algebra. Note however that although the Jacobi identity

$$[[A, B], C] + [[B, C], A] + [[C, A], B] = 0 \quad (27)$$

is satisfied when A , B and C belong to \mathcal{A} , the commutators of I with H and T_{\checkmark} cannot be written in terms of linear combinations of the generators of \mathcal{A} . Hence, \mathcal{A} is an enveloping algebra rather than a Lie algebra.

In quantum mechanics, we almost always use Lie algebras. If we insisted that the algebra of observables be a Lie algebra, we should modify (26) accordingly. The simplest Lie algebra that includes T_{\checkmark} is

$$\mathcal{A}' = \{T_{\checkmark}^2, T, H, 1\}. \quad (28)$$

One can check that \mathcal{A}' is indeed a Lie algebra. The price to pay, however, is that we are forced to include the non-selfadjoint operator T and that we cannot include T_{\checkmark} but T_{\checkmark}^2 .

Thus, we can either include T_{\checkmark} in a non-enveloping algebra or we can include T_{\checkmark}^2 and the non-selfadjoint T in a Lie algebra. Further progress must be made to see if the limitations of these algebras can be somehow overcome.

4 Conclusion

In order to summarize the results of this paper, we compare the properties of T with those of T_{\checkmark} . Whereas the operator T is not selfadjoint and its spectrum is the closed, upper half of the complex plane, the operator T_{\checkmark} is selfadjoint and its spectrum is the positive real line. There is a time representation associated with the operator T_{\checkmark} on which T_{\checkmark} acts as multiplication by t , whereas the POVM associated with T does not provide a well-defined time representation. Whereas T commutes canonically with H , T_{\checkmark} does not.

Finally, we note that by applying the HFT to the radial momentum operator in three dimensions and to the momentum operator in the half-line, we can also construct selfadjoint variants of these operators.

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